Uniform-in-time propagation of chaos for mean-field Langevin dynamics with convex energy

Songbo Wang

CMAP, École polytechnique

Séminaire des doctorants de FIME, 2023/02/17

Joint work with F. Chen and Z. Ren

Table of Contents

1. Review of linear Langevin dynamics
2. Mean-field Langevin dynamics
3. Propagation of chaos

Classical Langevin dynamics

Let $U : \mathbb{R}^d \to \mathbb{R}$ be the potential.

Classical Langevin dynamics

Let $U : \mathbb{R}^d \to \mathbb{R}$ be the potential. We are interested in the "overdamped" Langevin dynamics in $\mathbb{R}^d$

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t,$$

Classical Langevin dynamics

Let $U : \mathbb{R}^d \to \mathbb{R}$ be the potential. We are interested in the "overdamped" Langevin dynamics in $\mathbb{R}^d$

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t,$$

and the "underdamped" (kinetic) one

$$\begin{aligned} dX_t &= \alpha V_t dt, \\ dV_t &= -(\gamma V_t + \nabla U(X_t))dt + \sigma dW_t. \end{aligned}$$

Classical Langevin dynamics

Let $U : \mathbb{R}^d \to \mathbb{R}$ be the potential. We are interested in the "overdamped" Langevin dynamics in $\mathbb{R}^d$

$$dX_t = -\nabla U(X_t)dt + \sigma dW_t,$$

and the "underdamped" (kinetic) one

$$\begin{aligned} dX_t &= \alpha V_t dt, \\ dV_t &= -(\gamma V_t + \nabla U(X_t))dt + \sigma dW_t. \end{aligned}$$

Scaling transformation on (X, t) ((X, V, t) resp.) allows one to fix constants

$$\sigma = \sqrt{2}$$
$$(\alpha, \gamma, \sigma) = (1, 1, \sqrt{2}), \text{ resp.}$$

upon a redefinition of U.

Fokker–Planck equation

In the overdamped case, the probability density of $\mathrm{Law}(X_t)$ solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla U) \text{ on } \mathbb{R}_+ \times \mathbb{R}^d$$

Fokker–Planck equation

In the overdamped case, the probability density of $\mathrm{Law}(X_t)$ solves

$$\partial_t \rho = \Delta\rho + \nabla \cdot (\rho \nabla U) \text{ on } \mathbb{R}_+ \times \mathbb{R}^d$$

with stationary solution

$$\rho_\infty(x) \propto \exp(-U(x))$$

Fokker–Planck equation

In the overdamped case, the probability density of $\mathrm{Law}(X_t)$ solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla U) \text{ on } \mathbb{R}_+ \times \mathbb{R}^d$$

with stationary solution

$$\rho_\infty(x) \propto \exp(-U(x))$$

In the underdamped case, the probability density of $\mathrm{Law}(X_t, V_t)$ solves

$$\partial_t \rho = \Delta_v \rho + \nabla_v \cdot (\rho v) + \nabla U \cdot \nabla_v \rho - v \cdot \nabla_x \rho \text{ on } \mathbb{R}_+ \times \mathbb{R}^d_x \times \mathbb{R}^d_v$$

Fokker–Planck equation

In the overdamped case, the probability density of $\mathrm{Law}(X_t)$ solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla U) \text{ on } \mathbb{R}_+ \times \mathbb{R}^d$$

with stationary solution

$$\rho_\infty(x) \propto \exp(-U(x))$$

In the underdamped case, the probability density of $\mathrm{Law}(X_t, V_t)$ solves

$$\partial_t \rho = \Delta_v \rho + \nabla_v \cdot (\rho v) + \nabla U \cdot \nabla_v \rho - v \cdot \nabla_x \rho \text{ on } \mathbb{R}_+ \times \mathbb{R}^d_x \times \mathbb{R}^d_v$$

with stationary solution

$$\rho_\infty(x, v) \propto \exp\left(-U(x) - \frac{1}{2} v^2\right)$$

Free energy

In the overdamped case, define the free energy

$$F(\rho) = \text{energy} + \text{entropy} = \int \rho U + \int \rho \log \rho$$

Free energy

In the overdamped case, define the free energy

$$F(\rho) = \text{energy} + \text{entropy} = \int \rho U + \int \rho \log \rho$$

Recall the formula for Helmholtz free energy $F = U - TS$ with convention $S = -k_{\mathrm{B}} \int \rho \log \rho$.

Free energy

In the overdamped case, define the free energy

$$F(\rho) = \text{energy} + \text{entropy} = \int \rho U + \int \rho \log \rho$$

Recall the formula for Helmholtz free energy $F = U - TS$ with convention $S = -k_{\mathrm{B}} \int \rho \log \rho$.
Formal calculus, taking into account $\int \delta\rho = 0$, gives

$$\delta F(\rho) = \int (U + \log \rho + 1)\delta\rho = \int (U + \log \rho)\delta\rho$$

Free energy

In the overdamped case, define the free energy

$$F(\rho) = \text{energy} + \text{entropy} = \int \rho U + \int \rho \log \rho$$

Recall the formula for Helmholtz free energy $F = U - TS$ with convention $S = -k_{\mathrm{B}} \int \rho \log \rho$.
Formal calculus, taking into account $\int \delta\rho = 0$, gives

$$\delta F(\rho) = \int (U + \log \rho + 1)\delta\rho = \int (U + \log \rho)\delta\rho$$

So the minimizer should let $U + \log \rho = \text{constant}$, i.e. $\rho \propto \exp(-U(x))$.

Free energy as relative entropy

Recall $\rho_\infty = Z^{-1} \exp(-U(x))$, $Z = \int \exp(-U(x))$.

Free energy as relative entropy

Recall $\rho_\infty = Z^{-1} \exp(-U(x))$, $Z = \int \exp(-U(x))$. Rewrite the free energy

$$F(\rho) = \int (\log \rho + U)\rho = \int \left(\log \frac{\rho}{\rho_\infty} + \log Z \right) \rho = H(\rho|\rho_\infty) + \log Z$$

Free energy as relative entropy

Recall $\rho_\infty = Z^{-1}\exp(-U(x))$, $Z = \int \exp(-U(x))$. Rewrite the free energy

$$F(\rho) = \int (\log \rho + U)\rho = \int \left(\log \frac{\rho}{\rho_\infty} + \log Z \right) \rho = H(\rho|\rho_\infty) + \log Z$$

Plug in $\rho = \rho_\infty$, we have $F(\rho_\infty) = \log Z$.

Free energy as relative entropy

Recall $\rho_\infty = Z^{-1}\exp(-U(x))$, $Z = \int \exp(-U(x))$. Rewrite the free energy

$$F(\rho) = \int (\log\rho + U)\rho = \int \left(\log\frac{\rho}{\rho_\infty} + \log Z\right)\rho = H(\rho|\rho_\infty) + \log Z$$

Plug in $\rho = \rho_\infty$, we have $F(\rho_\infty) = \log Z$.
Recall formula $F = k_\mathrm{B} T \log Z$.

Free energy descent

Now compute the change of free energy along the Langevin dynamics.

Free energy descent

Now compute the change of free energy along the Langevin dynamics. Recall ρ_t solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla U) = \nabla \cdot (\rho(\nabla U + \nabla \log \rho))$$

Free energy descent

Now compute the change of free energy along the Langevin dynamics. Recall ρ_t solves

$$\partial_t \rho = \Delta \rho + \nabla \cdot (\rho \nabla U) = \nabla \cdot (\rho(\nabla U + \nabla \log \rho))$$

Hence

$$\begin{aligned}
\frac{d}{dt} H(\rho_t|\rho_\infty) = \frac{dF(\rho_t)}{dt} &= \frac{d}{dt} \int \rho_t U + \rho_t \log \rho_t \\
&= \int (U + \log \rho_t) \partial_t \rho_t \\
&= \int (U + \log \rho_t) \nabla \cdot (\rho_t \nabla (U + \log \rho_t)) \\
&= \int \rho_t |\nabla (U + \log \rho_t)|^2 = \int \left| \nabla \log \frac{\rho_t}{\rho_\infty} \right|^2 \rho_t
\end{aligned}$$

Logarithmic Sobolev inequality

Define the relative Fisher information $I(\rho_1|\rho_2) = \int |\nabla \log \frac{\rho_1}{\rho_2}|^2 \rho_1$.

Logarithmic Sobolev inequality

Define the relative Fisher information $I(\rho_1|\rho_2) = \int |\nabla \log \frac{\rho_1}{\rho_2}|^2 \rho_1$.
Free energy descent writes $\frac{d}{dt} H(\rho|\rho_\infty) = -I(\rho|\rho_\infty)$.

Logarithmic Sobolev inequality

Define the relative Fisher information $I(\rho_1|\rho_2) = \int |\nabla \log \frac{\rho_1}{\rho_2}|^2 \rho_1$.
Free energy descent writes $\frac{d}{dt} H(\rho|\rho_\infty) = -I(\rho|\rho_\infty)$.
Note $H, I = 0$ iff two measures are identical.

Logarithmic Sobolev inequality

Define the relative Fisher information $I(\rho_1|\rho_2) = \int |\nabla \log \frac{\rho_1}{\rho_2}|^2 \rho_1$.
Free energy descent writes $\frac{d}{dt} H(\rho|\rho_\infty) = -I(\rho|\rho_\infty)$.
Note $H, I = 0$ iff two measures are identical.
We say $\rho_\infty \propto \exp(-U)$ satisfies the κ-LSI if for any measure ρ, we have

$$I(\rho|\rho_\infty) \geq 2\kappa H(\rho|\rho_\infty).$$

Logarithmic Sobolev inequality

Define the relative Fisher information $I(\rho_1|\rho_2) = \int |\nabla \log \frac{\rho_1}{\rho_2}|^2 \rho_1$.
Free energy descent writes $\frac{d}{dt} H(\rho|\rho_\infty) = -I(\rho|\rho_\infty)$.
Note $H, I = 0$ iff two measures are identical.
We say $\rho_\infty \propto \exp(-U)$ satisfies the κ-LSI if for any measure ρ, we have

$$I(\rho|\rho_\infty) \geq 2\kappa H(\rho|\rho_\infty).$$

Thus if ρ_∞ satisfies κ-LSI, then we have the exponential convergence of free energy (relative entropy)

$$H(\rho_t|\rho_\infty) \leq H(\rho_0|\rho_\infty) e^{-2\kappa t}.$$

Conditions for LSI

(Bakry–Émery) If $\nabla^2 U \geq \kappa > 0$, then $\rho_\infty \propto \exp(-U)$ satisfies κ-LSI.

Conditions for LSI

(Bakry–Émery) If $\nabla^2 U \geq \kappa > 0$, then $\rho_\infty \propto \exp(-U)$ satisfies κ-LSI.
Caveat: Otto and Villani use Langevin to prove LSI, so it's a logical loop.

Conditions for LSI

(Bakry–Émery) If $\nabla^2 U \geq \kappa > 0$, then $\rho_\infty \propto \exp(-U)$ satisfies κ-LSI.
Caveat: Otto and Villani use Langevin to prove LSI, so it's a logical loop.
(Holley–Stroock) If ρ satisfies an LSI, and $\log \frac{\rho'}{\rho}$ is bounded, then ρ' satisfies an LSI with a weaker constant.

Conditions for LSI

(Bakry–Émery) If $\nabla^2 U \geq \kappa > 0$, then $\rho_\infty \propto \exp(-U)$ satisfies κ-LSI.
Caveat: Otto and Villani use Langevin to prove LSI, so it's a logical loop.
(Holley–Stroock) If ρ satisfies an LSI, and $\log \frac{\rho'}{\rho}$ is bounded, then ρ' satisfies an LSI with a weaker constant.
(Aida–Shigekawa) If ρ satisfies an LSI, and $\log \frac{\rho'}{\rho}$ is Lipschitz continuous, then ρ' satisfies an LSI with a weaker constant.

Conditions for LSI

(Bakry–Émery) If $\nabla^2 U \geq \kappa > 0$, then $\rho_\infty \propto \exp(-U)$ satisfies κ-LSI.
Caveat: Otto and Villani use Langevin to prove LSI, so it's a logical loop.
(Holley–Stroock) If ρ satisfies an LSI, and $\log \frac{\rho'}{\rho}$ is bounded, then ρ' satisfies an LSI with a weaker constant.
(Aida–Shigekawa) If ρ satisfies an LSI, and $\log \frac{\rho'}{\rho}$ is Lipschitz continuous, then ρ' satisfies an LSI with a weaker constant.
These conditions are all dimension-free!

Underdamped case

Free energy $F(\rho) = \int \rho(U + \frac{1}{2}v^2) + \rho \log \rho$.

Underdamped case

Free energy $F(\rho) = \int \rho(U + \frac{1}{2}v^2) + \rho \log \rho$.
Still true:

- $\rho_\infty = Z^{-1} \exp(-U - \frac{1}{2}v^2), Z = \int \exp(-U - \frac{1}{2}v^2)$;
- $F(\rho) = H(\rho|\rho_\infty) + \log Z$.

Underdamped case

Free energy $F(\rho) = \int \rho(U + \frac{1}{2}v^2) + \rho \log \rho$.
Still true:

- $\rho_\infty = Z^{-1} \exp(-U - \frac{1}{2}v^2), Z = \int \exp(-U - \frac{1}{2}v^2)$;
- $F(\rho) = H(\rho|\rho_\infty) + \log Z$.

But free energy descent is no longer the same

$$\frac{d}{dt} H(\rho|\rho_\infty) = -\int \left| \nabla_v \log \frac{\rho}{\rho_\infty} \right|^2 \rho$$

which is NOT the full Fisher information.

Underdamped case

Free energy $F(\rho) = \int \rho(U + \frac{1}{2}v^2) + \rho \log \rho$.
Still true:

- $\rho_\infty = Z^{-1} \exp(-U - \frac{1}{2}v^2), Z = \int \exp(-U - \frac{1}{2}v^2)$;
- $F(\rho) = H(\rho|\rho_\infty) + \log Z$.

But free energy descent is no longer the same

$$\frac{d}{dt} H(\rho|\rho_\infty) = -\int \left| \nabla_v \log \frac{\rho}{\rho_\infty} \right|^2 \rho$$

which is NOT the full Fisher information.
Free energy decrease vanishes for "local equilibrium states", that is, whose conditional law $\rho(v|x) \propto \exp(-\frac{1}{2}v^2)$.

Hypocoercivity

Nevertheless, Villani overcomes the problem by adding a term to relative entropy

$$\mathcal{E}(\rho) = H(\rho|\rho_\infty) + I_S(\rho|\rho_\infty)$$

where S is a non-degenerate $2d \times 2d$ matrix and

$$I_S(\rho|\rho_\infty) = \int \left(\nabla \log \frac{\rho}{\rho_\infty} \right)^t S \nabla \log \frac{\rho}{\rho_\infty} \rho.$$

If ρ_∞ satisfies an LSI and $\nabla^2 U$ is bounded, we can construct S such that

$$\frac{d}{dt}\mathcal{E}(\rho) \leq -\kappa' \mathcal{E}(\rho).$$

Non-linear energy functional

In classical Langevin, energy is always linear to the measure

$$\int \rho U \text{ or } \int \rho \left(U + \frac{1}{2} v^2 \right)$$

Non-linear energy functional

In classical Langevin, energy is always linear to the measure

$$\int \rho U \text{ or } \int \rho \left(U + \frac{1}{2} v^2 \right)$$

We consider mean-field interaction $E : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$.

Non-linear energy functional

In classical Langevin, energy is always linear to the measure

$$\int \rho U \text{ or } \int \rho \left(U + \frac{1}{2} v^2 \right)$$

We consider mean-field interaction $E : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$. Examples:

- $E(\rho) = \int \rho U$;

Non-linear energy functional

In classical Langevin, energy is always linear to the measure

$$\int \rho U \text{ or } \int \rho \left(U + \frac{1}{2} v^2 \right)$$

We consider mean-field interaction $E : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \cup \{+\infty\}$. Examples:

- $E(\rho) = \int \rho U$;
- $E(\rho) = \int \rho U_1 + \frac{1}{2} \iint U_2(x - y)\rho(dx)\rho(dy)$;
- loss function of neural networks.

Linear functional derivative

We say that $E : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a continuous mapping $\delta_\rho E : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that for all $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, it holds

$$E(\rho_1) - E(\rho_0) = \int_0^1 \int \delta_\rho E(\rho_t, x)(\rho_1 - \rho_0)(dx)dt$$

where $\rho_t = (1-t)\rho_0 + t\rho_1$.

Linear functional derivative

We say that $E : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a continuous mapping $\delta_\rho E : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that for all $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, it holds

$$E(\rho_1) - E(\rho_0) = \int_0^1 \int \delta_\rho E(\rho_t, x)(\rho_1 - \rho_0)(dx)dt$$

where $\rho_t = (1-t)\rho_0 + t\rho_1$.
Examples:

- $[\delta_\rho \int \rho U](\rho, x) = U(x)$;

Linear functional derivative

We say that $E : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is C^1 if there exists a continuous mapping $\delta_\rho E : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ such that for all $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$, it holds

$$E(\rho_1) - E(\rho_0) = \int_0^1 \int \delta_\rho E(\rho_t, x)(\rho_1 - \rho_0)(dx)dt$$

where $\rho_t = (1-t)\rho_0 + t\rho_1$.
Examples:

- $[\delta_\rho \int \rho U](\rho, x) = U(x)$;
- $[\delta_\rho \frac{1}{2} \iint U_2(x-y)\rho(dx)\rho(dy)](\rho, x) = \int U_2(x-y)\rho(dy)$;

Intrinsic derivative

The LFD $\delta_\rho E$ is defined up to a constant.
If $\delta_\rho E(\rho, x)$ satisfies the definition, then $\delta_\rho E(\rho, x) + G(\rho)$ also does.

Intrinsic derivative

The LFD $\delta_\rho E$ is defined up to a constant.
If $\delta_\rho E(\rho, x)$ satisfies the definition, then $\delta_\rho E(\rho, x) + G(\rho)$ also does.
To get rid of ambiguity, we define the intrinsic derivative (if it exists)

$$D_\rho E(\rho, x) = \nabla_x \delta_\rho E(\rho, x)$$

Intrinsic derivative

The LFD $\delta_\rho E$ is defined up to a constant.
If $\delta_\rho E(\rho, x)$ satisfies the definition, then $\delta_\rho E(\rho, x) + G(\rho)$ also does.
To get rid of ambiguity, we define the intrinsic derivative (if it exists)

$$D_\rho E(\rho, x) = \nabla_x \delta_\rho E(\rho, x)$$

If ρ_t is the flow of measure induced by the exponential of a vector field:

$$\frac{dX_t(x)}{dt} = v(X_t(x)), \quad X_0(x) = x,$$
$$\rho_t = (X_t)_\# \rho_0$$

Then ρ solves the continuity equation $\partial_t \rho = -\nabla \cdot (\rho v)$.

Intrinsic derivative

The LFD $\delta_\rho E$ is defined up to a constant.
If $\delta_\rho E(\rho, x)$ satisfies the definition, then $\delta_\rho E(\rho, x) + G(\rho)$ also does.
To get rid of ambiguity, we define the intrinsic derivative (if it exists)

$$D_\rho E(\rho, x) = \nabla_x \delta_\rho E(\rho, x)$$

If ρ_t is the flow of measure induced by the exponential of a vector field:

$$\frac{dX_t(x)}{dt} = v(X_t(x)), \quad X_0(x) = x,$$
$$\rho_t = (X_t)_\# \rho_0$$

Then ρ solves the continuity equation $\partial_t \rho = -\nabla \cdot (\rho v)$.
The change of E is $\frac{d}{dt} E(\rho_t) = \int \delta_\rho E(\rho_t) - \nabla \cdot (\rho v) = \int D_\rho E \cdot v \rho$.

Convexity of energy functional

We say E is convex if $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$,
$\rho_t = (1-t)\rho_0 + t\rho_t$ the functional (flat) interpolation.
Tangent inequality: $E(\rho_1) - E(\rho_0) \geq \int \delta_\rho E(\rho_0, x)(\rho_1 - \rho_0)(dx)$.

Convexity of energy functional

We say E is convex if $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$,
$\rho_t = (1-t)\rho_0 + t\rho_t$ the functional (flat) interpolation.
Tangent inequality: $E(\rho_1) - E(\rho_0) \geq \int \delta_\rho E(\rho_0, x)(\rho_1 - \rho_0)(dx)$.
Examples:

- $E(\rho) = \int \rho U$ (equality);

Convexity of energy functional

We say E is convex if $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$,
$\rho_t = (1-t)\rho_0 + t\rho_t$ the functional (flat) interpolation.
Tangent inequality: $E(\rho_1) - E(\rho_0) \geq \int \delta_\rho E(\rho_0, x)(\rho_1 - \rho_0)(dx)$.
Examples:

- $E(\rho) = \int \rho U$ (equality);
- $E(\rho) = \frac{1}{2} \int U_2(x-y)\rho(dx)\rho(dy)$ where all Fourier coefficients $\hat{U}_2(k) \geq 0$, e.g. Coulumb interaction;

Convexity of energy functional

We say E is convex if $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$,
$\rho_t = (1-t)\rho_0 + t\rho_t$ the functional (flat) interpolation.
Tangent inequality: $E(\rho_1) - E(\rho_0) \geq \int \delta_\rho E(\rho_0, x)(\rho_1 - \rho_0)(dx)$.
Examples:

- $E(\rho) = \int \rho U$ (equality);
- $E(\rho) = \frac{1}{2} \int U_2(x-y)\rho(dx)\rho(dy)$ where all Fourier coefficients $\hat{U}_2(k) \geq 0$, e.g. Coulumb interaction;
- neural networks with only one hidden layer.

Convexity of energy functional

We say E is convex if $E(\rho_t) \leq (1-t)E(\rho_0) + tE(\rho_1)$,
$\rho_t = (1-t)\rho_0 + t\rho_t$ the functional (flat) interpolation.
Tangent inequality: $E(\rho_1) - E(\rho_0) \geq \int \delta_\rho E(\rho_0, x)(\rho_1 - \rho_0)(dx)$.
Examples:

- $E(\rho) = \int \rho U$ (equality);
- $E(\rho) = \frac{1}{2} \int U_2(x-y)\rho(dx)\rho(dy)$ where all Fourier coefficients $\hat{U}_2(k) \geq 0$, e.g. Coulumb interaction;
- neural networks with only one hidden layer.

Our key assumption is the functional convexity of E.

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \mathrm{Law}(X_t)$.

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \mathrm{Law}(X_t)$. In the underdamped case, we consider

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -(V_t + D_\rho E(\rho_t, X_t))dt + \sqrt{2}dW_t \end{aligned}$$

where $\rho_t = \mathrm{Law}(X_t)$

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \text{Law}(X_t)$. In the underdamped case, we consider

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -(V_t + D_\rho E(\rho_t, X_t))dt + \sqrt{2}dW_t \end{aligned}$$

where $\rho_t = \text{Law}(X_t)$
Classical case $D_\rho E(\rho, x) = \nabla U(x)$.

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \mathrm{Law}(X_t)$. In the underdamped case, we consider

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -(V_t + D_\rho E(\rho_t, X_t))dt + \sqrt{2}dW_t \end{aligned}$$

where $\rho_t = \mathrm{Law}(X_t)$
Classical case $D_\rho E(\rho, x) = \nabla U(x)$.
We still define free energy $F(\rho) = E(\rho) + \int \rho \log \rho$.

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \mathrm{Law}(X_t)$. In the underdamped case, we consider

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -(V_t + D_\rho E(\rho_t, X_t))dt + \sqrt{2}dW_t \end{aligned}$$

where $\rho_t = \mathrm{Law}(X_t)$
Classical case $D_\rho E(\rho, x) = \nabla U(x)$.
We still define free energy $F(\rho) = E(\rho) + \int \rho \log \rho$.
Classical overdamped Langevin is gradient descent in sense of JKO:

$$\rho^h_{n+1} = \mathrm{argmin} \int \rho(U + \log \rho) + (2h)^{-1} W_2^2(\rho, \rho^h_n),$$

$$\lim_{nh \to t, n \to \infty} \rho^h_n = \rho_t$$

Mean-field Langevin dynamics

In the overdamped case, we consider

$$dX_t = -D_\rho E(\rho_t, X_t)dt + \sqrt{2}dW_t$$

where $\rho_t = \mathrm{Law}(X_t)$. In the underdamped case, we consider

$$\begin{aligned} dX_t &= V_t dt \\ dV_t &= -(V_t + D_\rho E(\rho_t, X_t))dt + \sqrt{2}dW_t \end{aligned}$$

where $\rho_t = \mathrm{Law}(X_t)$
Classical case $D_\rho E(\rho, x) = \nabla U(x)$.
We still define free energy $F(\rho) = E(\rho) + \int \rho \log \rho$.
Overdamped MFL is its generalization

$$\rho^h_{n+1} = \mathrm{argmin}\, F(\rho) + (2h)^{-1} W_2^2(\rho, \rho^h_n),$$

$$\lim_{nh \to t, n \to \infty} \rho^h_n = \rho_t$$

MFL energy descent (1)

Same calculus as linear shows

$$\frac{d}{dt}\left(E(\rho_t) + \int \rho_t \log \rho_t\right) = \frac{d}{dt}F(\rho_t) = -I(\rho_t|\hat{\rho}_t)$$

where $\hat{\rho}_t(x) \propto \exp(-\delta_\rho E(\rho_t, x))$,

MFL energy descent (1)

Same calculus as linear shows

$$\frac{d}{dt}\left(E(\rho_t) + \int \rho_t \log \rho_t\right) = \frac{d}{dt}F(\rho_t) = -I(\rho_t|\hat{\rho}_t)$$

where $\hat{\rho}_t(x) \propto \exp(-\delta_\rho E(\rho_t, x))$, which is the Gibbs measure of a single particle as if all the others are frozen.

MFL energy descent (1)

Same calculus as linear shows

$$\frac{d}{dt}\left(E(\rho_t) + \int \rho_t \log \rho_t\right) = \frac{d}{dt}F(\rho_t) = -I(\rho_t|\hat{\rho}_t)$$

where $\hat{\rho}_t(x) \propto \exp(-\delta_\rho E(\rho_t, x))$, which is the Gibbs measure of a single particle as if all the others are frozen.
By convexity and standard arguments there exists unique minimizer of F, satisfying the fixed point condition $\rho_\infty = \hat{\rho}_\infty \propto \exp(-\delta_\rho E(\rho_t, \cdot))$.

MFL energy descent (1)

Same calculus as linear shows

$$\frac{d}{dt}\left(E(\rho_t) + \int \rho_t \log \rho_t\right) = \frac{d}{dt}F(\rho_t) = -I(\rho_t|\hat{\rho}_t)$$

where $\hat{\rho}_t(x) \propto \exp(-\delta_\rho E(\rho_t, x))$, which is the Gibbs measure of a single particle as if all the others are frozen.

By convexity and standard arguments there exists unique minimizer of F, satisfying the fixed point condition $\rho_\infty = \hat{\rho}_\infty \propto \exp(-\delta_\rho E(\rho_t, \cdot))$. We wish to show that $F(\rho_t) - F(\rho_\infty)$ converges exponentially.

MFL energy descent (1)

Same calculus as linear shows

$$\frac{d}{dt}\left(E(\rho_t) + \int \rho_t \log \rho_t\right) = \frac{d}{dt}F(\rho_t) = -I(\rho_t|\hat{\rho}_t)$$

where $\hat{\rho}_t(x) \propto \exp(-\delta_\rho E(\rho_t, x))$, which is the Gibbs measure of a single particle as if all the others are frozen.

By convexity and standard arguments there exists unique minimizer of F, satisfying the fixed point condition $\rho_\infty = \hat{\rho}_\infty \propto \exp(-\delta_\rho E(\rho_t, \cdot))$. We wish to show that $F(\rho_t) - F(\rho_\infty)$ converges exponentially.

Note that

$$\begin{aligned} F(\rho_t) - F(\rho_\infty) &\geq \int \delta_\rho E(\rho_\infty, \cdot)(\rho_t - \rho_\infty) + \int \rho_t \log \rho_t - \rho_\infty \log \rho_\infty \\ &= -\int \log \rho_\infty(\rho_t - \rho_\infty) + \int \rho_t \log \rho_t - \rho_\infty \log \rho_\infty \\ &= H(\rho_t|\rho_\infty). \end{aligned}$$

MFL energy descent (2)

We suppose $\hat{\rho}_t \propto \exp(-\delta_\rho E(\rho_t, \cdot))$ satisfies uniform LSI.

MFL energy descent (2)

We suppose $\hat{\rho}_t \propto \exp(-\delta_\rho E(\rho_t, \cdot))$ satisfies uniform LSI.
For example, suppose the LFD is of form $\delta_\rho E(\rho, x) = g(x) + h(\rho, x)$, where $\nabla^2 g \geq \kappa > 0$ and $h(\rho, x)$ be uniformly bounded. Then apply Bakry–Émery and Holley–Stroock.

MFL energy descent (2)

We suppose $\hat{\rho}_t \propto \exp(-\delta_\rho E(\rho_t, \cdot))$ satisfies uniform LSI.
For example, suppose the LFD is of form $\delta_\rho E(\rho, x) = g(x) + h(\rho, x)$, where $\nabla^2 g \geq \kappa > 0$ and $h(\rho, x)$ be uniformly bounded. Then apply Bakry–Émery and Holley–Stroock.
Let κ' be the uniform LSI constant.

$$\begin{aligned}
\frac{1}{2\kappa'} I(\rho_t|\hat{\rho}_t) &\geq H(\rho_t|\hat{\rho}_t) \\
&= \int \rho_t(\log \rho_t + \delta_\rho E(\rho_t, \cdot)) + \log \int \exp(-\delta_\rho E(\rho_t, \cdot)) \\
&= \int \rho_t(\log \rho_t + \delta_\rho E(\rho_t, \cdot)) + \log \int \exp(-\delta_\rho E(\rho_t, \cdot) - \log \rho_\infty)\rho_\infty \\
&\geq \int \rho_t(\log \rho_t + \delta_\rho E(\rho_t, \cdot)) - \int (\delta_\rho E(\rho_t, \cdot) + \log \rho_\infty)\rho_\infty \\
&= \int \delta_\rho E(\rho_t, \cdot)(\rho_t - \rho_\infty) + H(\rho_t) - H(\rho_\infty) \geq F(\rho_t) - F(\rho_\infty).
\end{aligned}$$

Hypocoercivity with mean-field interaction

À la Villani, we consider the functional

$$\mathcal{E}(\rho) = E(\rho) + \frac{1}{2}\int \rho v^2 + \int \rho \log \rho + I_S(\rho|\hat{\rho}).$$

Here $\hat{\rho} \propto \exp\left(-\delta_\rho E(\rho, x) - \frac{1}{2}v^2\right)$.

Hypocoercivity with mean-field interaction

À la Villani, we consider the functional

$$\mathcal{E}(\rho) = E(\rho) + \frac{1}{2}\int \rho v^2 + \int \rho \log \rho + I_S(\rho|\hat{\rho}).$$

Here $\hat{\rho} \propto \exp\left(-\delta_\rho E(\rho, x) - \frac{1}{2}v^2\right)$.
If

- $\hat{\rho}_t$ has uniform LSI;
- $\nabla D_\rho E(\rho_t, x)$ uniformly bounded;
- $\int |D_\rho^2 E(\rho_t, x, y)|^2 \rho_t(dx)\rho_t(dy)$ uniformly bounded;

Hypocoercivity with mean-field interaction

À la Villani, we consider the functional

$$\mathcal{E}(\rho) = E(\rho) + \frac{1}{2}\int \rho v^2 + \int \rho \log \rho + I_S(\rho|\hat{\rho}).$$

Here $\hat{\rho} \propto \exp\left(-\delta_\rho E(\rho, x) - \frac{1}{2}v^2\right)$.
If

- $\hat{\rho}_t$ has uniform LSI;
- $\nabla D_\rho E(\rho_t, x)$ uniformly bounded;
- $\int |D_\rho^2 E(\rho_t, x, y)|^2 \rho_t(dx)\rho_t(dy)$ uniformly bounded;

we can construct non-degenerate S such that

$$\frac{d}{dt}\mathcal{E}(\rho_t) \le -\kappa'(\mathcal{E}(\rho_t) - \mathcal{E}(\rho_\infty)).$$

Particle approximation

We consider the overdamped particle system

$$dX_t^i = -D_\rho F(\mu^{X_t}, X_t^i)dt + \sqrt{2}dW_t^i$$

where $\mu^{X_t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure.

Particle approximation

We consider the overdamped particle system

$$dX_t^i = -D_\rho F(\mu^{X_t}, X_t^i)dt + \sqrt{2}dW_t^i$$

where $\mu^{X_t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure.
We hope to show that when $N \to \infty$, the joint $\rho_t^N = \mathrm{Law}(X_t^1, \ldots, X_t^N) \to \rho_t^{\otimes N}$, in some sense. This convergence is called *propagation of chaos*.

Particle approximation

We consider the overdamped particle system

$$dX_t^i = -D_\rho F(\mu^{X_t}, X_t^i)dt + \sqrt{2}dW_t^i$$

where $\mu^{X_t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure.
We hope to show that when $N \to \infty$, the joint $\rho_t^N = \mathrm{Law}(X_t^1, \ldots, X_t^N) \to \rho_t^{\otimes N}$, in some sense. This convergence is called *propagation of chaos*.
Similarly the underdamped system can be defined

$$\begin{aligned} dX_t^i &= V_t^i dt \\ dV_t^i &= -(V_t^i + D_\rho F(\mu^{X_t,V_t}, X_t^i))dt + \sqrt{2}dW_t^i \end{aligned}$$

Particle approximation

We consider the overdamped particle system

$$dX_t^i = -D_\rho F(\mu^{X_t}, X_t^i)dt + \sqrt{2}dW_t^i$$

where $\mu^{X_t} = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}$ is the empirical measure.
We hope to show that when $N \to \infty$, the joint $\rho_t^N = \mathrm{Law}(X_t^1, \ldots, X_t^N) \to \rho_t^{\otimes N}$, in some sense. This convergence is called *propagation of chaos*.
Similarly the underdamped system can be defined

$$\begin{aligned} dX_t^i &= V_t^i dt \\ dV_t^i &= -(V_t^i + D_\rho F(\mu^{X_t, V_t}, X_t^i))dt + \sqrt{2}dW_t^i \end{aligned}$$

and we can consider $\rho_t^N = \mathrm{Law}(X_t^i, V_t^i)_{i=1}^N \to \rho_t^{\otimes N}$.

Distances between probability measure

We wish to show *uniformly* in t.

$$\frac{1}{N} W_2^2(\rho_t^N, \rho_t^{\otimes N}), \frac{1}{N} H(\rho_t^N | \rho_t^{\otimes N}) \to 0$$

Distances between probability measure

We wish to show *uniformly* in t.

$$\frac{1}{N} W_2^2(\rho_t^N, \rho_t^{\otimes N}), \frac{1}{N} H(\rho_t^N | \rho_t^{\otimes N}) \to 0$$

The factor $\frac{1}{N}$ takes into the account the addivity w.r.t. tensorization

$$W_2^2(\mu \otimes \mu', \nu \otimes \nu') = W_2^2(\mu, \mu') + W_2^2(\nu, \nu')$$
$$H(\mu \otimes \mu' | \nu \otimes \nu') = H(\mu | \mu') + H(\nu | \nu')$$

Distances between probability measure

We wish to show *uniformly* in t.

$$\frac{1}{N} W_2^2(\rho_t^N, \rho_t^{\otimes N}), \frac{1}{N} H(\rho_t^N | \rho_t^{\otimes N}) \to 0$$

The factor $\frac{1}{N}$ takes into the account the addivity w.r.t. tensorization

$$\begin{aligned} W_2^2(\mu \otimes \mu', \nu \otimes \nu') &= W_2^2(\mu, \mu') + W_2^2(\nu, \nu') \\ H(\mu \otimes \mu' | \nu \otimes \nu') &= H(\mu|\mu') + H(\nu|\nu') \end{aligned}$$

Synchronized coupling gives Grönwall-type growth estimate

$$\frac{1}{N} W_2^2(\rho_t^N, \rho_t^{\otimes N}) \leq \frac{1}{N} \exp(Ct) W_2^2(\rho_0^N, \rho_0^{\otimes N}) + \frac{C}{N}(\exp(Ct) - 1).$$

Distances between probability measure

We wish to show *uniformly* in t.

$$\frac{1}{N}W_2^2(\rho_t^N,\rho_t^{\otimes N}),\frac{1}{N}H(\rho_t^N|\rho_t^{\otimes N})\to 0$$

The factor $\frac{1}{N}$ takes into the account the addivity w.r.t. tensorization

$$W_2^2(\mu\otimes\mu',\nu\otimes\nu')=W_2^2(\mu,\mu')+W_2^2(\nu,\nu')$$
$$H(\mu\otimes\mu'|\nu\otimes\nu')=H(\mu|\mu')+H(\nu|\nu')$$

Synchronized coupling gives Grönwall-type growth estimate

$$\frac{1}{N}W_2^2(\rho_t^N,\rho_t^{\otimes N})\leq\frac{1}{N}\exp(Ct)W_2^2(\rho_0^N,\rho_0^{\otimes N})+\frac{C}{N}(\exp(Ct)-1).$$

We focus on the *long-time behavior* of the particle system.

Free energy descent for particle system (1)

We note that the N-particle system is subject to the potential in $\mathbb{R}^{Nd}$

$$U^N(x_1, \ldots, x_N) = NE\left(\frac{1}{N}\sum_{i=1}^{N}\delta_{x^i}\right) = NE(\mu^x)$$

$$\nabla_i U^N(x_1, \ldots, x_N) = D_\rho E\left(\frac{1}{N}\sum_{j=1}^{N}\delta_{x^j}, x_i\right) = D_\rho E(\mu^x, x^i).$$

Free energy descent for particle system (1)

We note that the N-particle system is subject to the potential in $\mathbb{R}^{Nd}$

$$U^N(x_1, \ldots, x_N) = NE\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}\right) = NE(\mu^x)$$

$$\nabla_i U^N(x_1, \ldots, x_N) = D_\rho E\left(\frac{1}{N}\sum_{j=1}^N \delta_{x^j}, x_i\right) = D_\rho E(\mu^x, x^i).$$

Define free energy $F^N(\rho^N) = N\int E(\mu^x)\rho^N(dx) + \int \rho^N \log \rho^N$ and the invariant measure $\rho^N_\infty \propto \exp(-NE(\mu^x))$. We know by classical results

$$\frac{d}{dt}F^N(\rho^N_t) = \frac{d}{dt}H(\rho^N_t|\rho^N_\infty) = -I(\rho^N_t|\rho^N_\infty).$$

Free energy descent for particle system (1)

We note that the N-particle system is subject to the potential in $\mathbb{R}^{Nd}$

$$U^N(x_1,\ldots,x_N) = NE\left(\frac{1}{N}\sum_{i=1}^N \delta_{x^i}\right) = NE(\mu^x)$$

$$\nabla_i U^N(x_1,\ldots,x_N) = D_\rho E\left(\frac{1}{N}\sum_{j=1}^N \delta_{x^j}, x_i\right) = D_\rho E(\mu^x, x^i).$$

Define free energy $F^N(\rho^N) = N\int E(\mu^x)\rho^N(dx) + \int \rho^N \log \rho^N$ and the invariant measure $\rho^N_\infty \propto \exp(-NE(\mu^x))$. We know by classical results

$$\frac{d}{dt}F^N(\rho^N_t) = \frac{d}{dt}H(\rho^N_t|\rho^N_\infty) = -I(\rho^N_t|\rho^N_\infty).$$

However, under our conditions the LSI constant for ρ^N_∞ vanishes when $N\to\infty$.

Free energy descent for particle system (2)

Our idea is to decompose $I(\rho_t^N|\rho_\infty^N)$ componentwise.

Free energy descent for particle system (2)

Our idea is to decompose $I(\rho_t^N|\rho_\infty^N)$ componentwise. Denote $-i = \{1, \dots, n\}\backslash\{i\}$ and for $\rho^N \in \mathcal{P}(\mathbb{R}^{Nd})$, denote

- the marginal distribution by $\rho^N_{i_1,\dots,i_k}(x_{i_1}, \dots, x_{i_k})$;
- the conditonal distribution by $\rho^N_{i|-i}(x_i|x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \rho^N_{i|-i}(x_i|x_{-i})$.

Free energy descent for particle system (2)

Our idea is to decompose $I(\rho_t^N|\rho_\infty^N)$ componentwise. Denote $-i = \{1,\ldots,n\}\backslash\{i\}$ and for $\rho^N \in \mathcal{P}(\mathbb{R}^{Nd})$, denote

- the marginal distribution by $\rho^N_{i_1,\ldots,i_k}(x_{i_1},\ldots,x_{i_k})$;
- the conditonal distribution by $\rho^N_{i|-i}(x_i|x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n) = \rho^N_{i|-i}(x_i|x_{-i})$.

Fisher information decomposes

$$I(\rho_t^N|\rho_\infty^N) = \sum_{i=1}^N \int \left| \nabla_i \log \frac{\rho^N_{i|-i,t}\rho^N_{-i,t}}{\rho^N_{i|-i,\infty}\rho^N_{-i,\infty}} \right|^2 \rho^N_{i|-i,t}\rho^N_{-i}$$

$$= \sum_{i=1}^N \mathbb{E}^{-i}\left[\int \left| D_\rho E(\mu^x, x^i) + \nabla_i \log \rho^N_{i|-i} \right|^2 \rho^N_{i|-i} \right]$$

Modulo an $o(1)$ error, we may replace μ^x by $\mu^{x_{-i}} = \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}$.

Free energy descent for particle system (3)

Hence $I(\rho_t^N|\rho_\infty^N) = \sum_{i=1}^N \mathbb{E}^{-i}\, I(\rho_{i|-i}^N|\hat\rho_{i|-i}^N) + o(N)$, with

$$\hat\rho_{i|-i}^N \propto \exp(-\delta_\rho E(\mu^{x_{-i}}, x^i)).$$

Free energy descent for particle system (3)

Hence $I(\rho_t^N|\rho_\infty^N) = \sum_{i=1}^N \mathbb{E}^{-i}\, I(\rho_{i|-i}^N|\hat\rho_{i|-i}^N) + o(N)$, with

$$\hat\rho_{i|-i}^N \propto \exp(-\delta_\rho E(\mu^{x_{-i}}, x^i)).$$

This is the equilibrium measure for i-th particle as if all the other particles are frozen.

Free energy descent for particle system (3)

Hence $I(\rho_t^N|\rho_\infty^N) = \sum_{i=1}^N \mathbb{E}^{-i}\, I(\rho_{i|-i}^N|\hat\rho_{i|-i}^N) + o(N)$, with

$$\hat\rho_{i|-i}^N \propto \exp(-\delta_\rho E(\mu^{x_{-i}}, x^i)).$$

This is the equilibrium measure for i-th particle as if all the other particles are frozen. The benefit is it satisfies LSI uniformly in N, i.

Free energy descent for particle system (3)

Hence $I(\rho_t^N|\rho_\infty^N) = \sum_{i=1}^N \mathbb{E}^{-i}\, I(\rho_{i|-i}^N|\hat\rho_{i|-i}^N) + o(N)$, with

$$\hat\rho_{i|-i}^N \propto \exp(-\delta_\rho E(\mu^{x_{-i}}, x^i)).$$

This is the equilibrium measure for i-th particle as if all the other particles are frozen. The benefit is it satisfies LSI uniformly in N, i.
Applying LSI and use Jensen's inequality as before

$$\begin{aligned}\frac{1}{2\kappa'}\sum_{i=1}^N \mathbb{E}^{-i}\, I(\rho_{i|-i,t}^N|\hat\rho_{i|-i}^N) &= \sum_{i=1}^N \mathbb{E}^{-i}\, H(\rho_{i|-i,t}^N|\hat\rho_{i|-i}^N) \\ &= \sum_{i=1}^N \mathbb{E}^{-i} \int (\rho_{i|-i,t}^N(\delta_\rho E(\mu^{x_{-i}}.x_i) + \log\rho_{i|-i,t}^N) + \log\int \exp(-\delta_\rho E(\mu^{x^i},\cdot)) \\ &\geq \sum_{i=1}^N \mathbb{E}^{-i}\int(\rho_{i|-i,t}^N - \rho_\infty)\delta_\rho E(\mu^{x_{-i}}.x_i) + \int \rho_{i|-i}^N\log\rho_{i|-i}^N - \rho_\infty\log\rho_\infty\end{aligned}$$

Free energy descent for particle system (4)

By convexity of E, we have

$$\begin{aligned} E^N(\rho_t^N) - NE(\rho_\infty) &= N\,\mathbb{E}[E(\mu^X) - E(\rho_\infty)] \\ &\le N\,\mathbb{E}[\int \delta_\rho E(\mu^X, x)(\mu^X - \rho_\infty)(dx)] \\ &= \mathbb{E}\left[\sum_{i=1}^N \delta_\rho E(\mu^X, X_{i,t}) - \int \delta_\rho E(\mu^X, x)\rho_\infty(dx)\right] \end{aligned}$$

Free energy descent for particle system (4)

By convexity of E, we have

$$\begin{aligned} E^N(\rho_t^N) - NE(\rho_\infty) &= N\,\mathbb{E}[E(\mu^X) - E(\rho_\infty)] \\ &\leq N\,\mathbb{E}[\int \delta_\rho E(\mu^X, x)(\mu^X - \rho_\infty)(dx)] \\ &= \mathbb{E}\left[\sum_{i=1}^N \delta_\rho E(\mu^X, X_{i,t}) - \int \delta_\rho E(\mu^X, x)\rho_\infty(dx)\right] \end{aligned}$$

Again changing μ^x to $\mu^{x_{-i}}$ and use the information inequality

$$\sum_{i=1}^N \mathbb{E}^{-i} \int \rho_{i|-i}^N \log \rho_{i|-i}^N \geq \int \rho^N \log \rho^N,$$

we derive

$$\frac{d}{dt} F^N(\rho_t^N) \leq -2\kappa'(F^N(\rho_t^N) - NF(\rho_\infty)) + o(N).$$

Free energy descent for particle system (5)

Previous differential inequality gives bound on energy of one side

$$\frac{1}{N}F^N(\rho_t^N) - F(\rho_\infty) \leq \left(\frac{1}{N}F^N(\rho_0^N) - F(\rho_\infty)\right) e^{-2\kappa' t} + o(1).$$

Free energy descent for particle system (5)

Previous differential inequality gives bound on energy of one side

$$\frac{1}{N}F^N(\rho_t^N) - F(\rho_\infty) \le \left(\frac{1}{N}F^N(\rho_0^N) - F(\rho_\infty)\right) e^{-2\kappa' t} + o(1).$$

The other bound is obtained by convexity

$$\begin{aligned}
\frac{1}{N}F^N(\rho_t^N) - F(\rho_\infty) &= \mathbb{E}[E(\mu^{X_t}) - E(\rho_\infty)] + \frac{1}{N}\int \rho_t^N \log \rho_t^N - \int \rho_\infty \log \rho_\infty \\
&\ge \mathbb{E}\left[\int \delta_\rho E(\rho_\infty, \cdot)(\mu^{X_t} - \rho_\infty)\right] + \frac{1}{N}\int \rho_t^N \log \rho_t^N - \int \rho_\infty \log \rho_\infty \\
&= \frac{1}{N}\int (\rho_t^N - \rho_\infty^{\otimes N}) \sum_{i=1}^N \delta_\rho E(\rho_\infty, x^i) + \frac{1}{N}\int \rho_t^N \log \rho_t^N - \rho_\infty^{\otimes N} \log \rho_\infty^{\otimes N} \\
&= \frac{1}{N} H(\rho_t^N | \rho_\infty^{\otimes N}) \ge 0.
\end{aligned}$$

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t} + o(1)$.

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t}$.

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \le C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \le C_0 e^{-2\kappa' t}$.
Since $\rho_\infty \propto \exp(-\delta_\rho E(\rho_\infty,\cdot))$ satisfies a LSI, $\rho_\infty^{\otimes N}$ too with the same constant.

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t}$.
Since $\rho_\infty \propto \exp(-\delta_\rho E(\rho_\infty, \cdot))$ satisfies a LSI, $\rho_\infty^{\otimes N}$ too with the same constant.
LSI implies Talagrand's T_2 inequality, $\frac{\kappa'}{2}W_2^2(\rho, \rho_\infty^{\otimes N}) \leq H(\rho|\rho_\infty^{\otimes N})$

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t}$.
Since $\rho_\infty \propto \exp(-\delta_\rho E(\rho_\infty, \cdot))$ satisfies a LSI, $\rho_\infty^{\otimes N}$ too with the same constant.
LSI implies Talagrand's T_2 inequality, $\frac{\kappa'}{2} W_2^2(\rho, \rho_\infty^{\otimes N}) \leq H(\rho|\rho_\infty^{\otimes N})$
Hence

$$\begin{aligned}\frac{1}{N}W_2^2(\rho_t^N, \rho_t^{\otimes N}) &\leq \frac{1}{N}W_2^2(\rho_t^N, \rho_\infty^{\otimes N}) + \frac{1}{N}W_2^2(\rho_t^{\otimes N}, \rho_\infty^{\otimes N}) \\ &\leq \frac{2}{\kappa'}\frac{1}{N}(H(\rho_t^N|\rho_t^{\otimes N}) + H(\rho_t^{\otimes N}|\rho_t^{\otimes N})) \leq \frac{C_0}{\kappa'}e^{-2\kappa' t} + o(1)\end{aligned}$$

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \le C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \le C_0 e^{-2\kappa' t}$.
Since $\rho_\infty \propto \exp(-\delta_\rho E(\rho_\infty,\cdot))$ satisfies a LSI, $\rho_\infty^{\otimes N}$ too with the same constant.
LSI implies Talagrand's T_2 inequality, $\frac{\kappa'}{2}W_2^2(\rho,\rho_\infty^{\otimes N}) \le H(\rho|\rho_\infty^{\otimes N})$
Hence

$$\begin{aligned}\frac{1}{N}W_2^2(\rho_t^N,\rho_t^{\otimes N}) &\le \frac{1}{N}W_2^2(\rho_t^N,\rho_\infty^{\otimes N}) + \frac{1}{N}W_2^2(\rho_t^{\otimes N},\rho_\infty^{\otimes N}) \\ &\le \frac{2}{\kappa'}\frac{1}{N}(H(\rho_t^N|\rho_t^{\otimes N}) + H(\rho_t^{\otimes N}|\rho_t^{\otimes N})) \le \frac{C_0}{\kappa'}e^{-2\kappa' t} + o(1)\end{aligned}$$

If E is regular enough, $o(1)$-term is actually $O(N^{-1})$.

Propagation of chaos in long time

From previous computations, $\frac{1}{N}H(\rho_t^N|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t} + o(1)$.
Tensorizing MFL $H(\rho_t|\rho_\infty)$, we have $\frac{1}{N}H(\rho_t^{\otimes N}|\rho_\infty^{\otimes N}) \leq C_0 e^{-2\kappa' t}$.
Since $\rho_\infty \propto \exp(-\delta_\rho E(\rho_\infty, \cdot))$ satisfies a LSI, $\rho_\infty^{\otimes N}$ too with the same constant.
LSI implies Talagrand's T_2 inequality, $\frac{\kappa'}{2} W_2^2(\rho, \rho_\infty^{\otimes N}) \leq H(\rho|\rho_\infty^{\otimes N})$
Hence

$$\begin{aligned}\frac{1}{N}W_2^2(\rho_t^N, \rho_t^{\otimes N}) &\leq \frac{1}{N}W_2^2(\rho_t^N, \rho_\infty^{\otimes N}) + \frac{1}{N}W_2^2(\rho_t^{\otimes N}, \rho_\infty^{\otimes N}) \\ &\leq \frac{2}{\kappa'}\frac{1}{N}(H(\rho_t^N|\rho_t^{\otimes N}) + H(\rho_t^{\otimes N}|\rho_t^{\otimes N})) \leq \frac{C_0}{\kappa'}e^{-2\kappa' t} + o(1)\end{aligned}$$

If E is regular enough, $o(1)$-term is actually $O(N^{-1})$.
Convergence of relative entropy requires uniform LSI for ρ_t.

Underdamped case

Still WIP, but method largely the same:

- Define a Lyapunov $\mathcal{E}^N$ for N-particle system;

Underdamped case

Still WIP, but method largely the same:

- Define a Lyapunov $\mathcal{E}^N$ for N-particle system;
- Compute hypocoercivity $\frac{d}{dt}\mathcal{E}^N \leq -\kappa' I_S^N$, uniformly in N;

Underdamped case

Still WIP, but method largely the same:

- Define a Lyapunov $\mathcal{E}^N$ for N-particle system;
- Compute hypocoercivity $\frac{d}{dt}\mathcal{E}^N \leq -\kappa' I_S^N$, uniformly in N;
- Use convexity and $\mu^x \leftrightarrow \mu^{x_{-i}}$ tricks before applying LSI;

Underdamped case

Still WIP, but method largely the same:

- Define a Lyapunov $\mathcal{E}^N$ for N-particle system;
- Compute hypocoercivity $\frac{d}{dt}\mathcal{E}^N \leq -\kappa' I_S^N$, uniformly in N;
- Use convexity and $\mu^x \leftrightarrow \mu^{x_{-i}}$ tricks before applying LSI;
- Bound relative entropy by free energy difference.

Underdamped case

Still WIP, but method largely the same:

- Define a Lyapunov $\mathcal{E}^N$ for N-particle system;
- Compute hypocoercivity $\frac{d}{dt}\mathcal{E}^N \leq -\kappa' I_S^N$, uniformly in N;
- Use convexity and $\mu^x \leftrightarrow \mu^{x_{-i}}$ tricks before applying LSI;
- Bound relative entropy by free energy difference.

The method for relative entropy's and *relative Fisher information's* convergence is different, and we lose some exponents in N.